Equation with 1 or 2 variables

Topic	Algorithms
Learning objectives	Know how to report points in a Cartesian coordinate system, solution of equations in two variables. Know how to solve maximum/minimum problems with quadratic functions. Know how to obtain the equations of conics from data and vice versa.
Age group	14-18 years (to be adapted in each country)
Estimated duration	2 h
Activities	Search for special points of ellipses and parabolas obtained from real situations
Related visits	Pisa, Lucca

Previous knowledge required

Know the Cartesian plane as a coordinate reference system; know how to recognize the type of conic when it is put into normal shape.

Step by step: the sequence in the classroom

Step 1: Introducing the topic

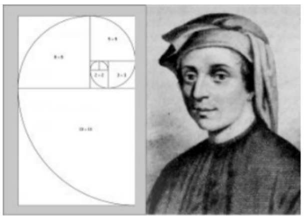
Equations with two unknowns is a topic that ranges from Analysis to Geometry and Algebra. They are equations in x and y and the solutions are all and only the ordered pairs (x;y) that make the equality true. The set of these solutions corresponds to a geometric place in the Cartesian plane. In particular, we will deal with the parabola and the ellipse.

Short presentation of the heritage elements in this sequence

Al-Khwarizmi-The father of Algebra-olympic.uz

The equations were introduced for the first time in the 9th century A.D. by the Persian mathematician astronomer Al-Khwarizmi to solve mercantile or inheritance division problems.

Classical algebra, however, was born and developed in the Arab world; in Italy, we had to wait until the 11th century, when the Abacus schools arose, to which Leonardo Pisano, known as Fibonacci, made a notable contribution.



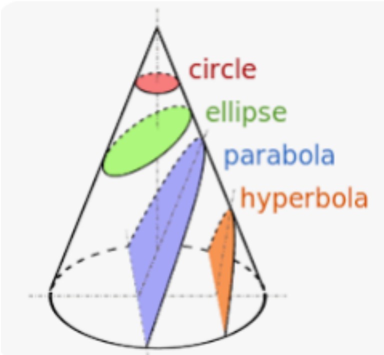
Leonardo Pisano-La tecnica della scuola

Later, in the Renaissance, there were great algebraists such as Pacioli, Cardano and Bombelli.

Only later, in the 17th century, symbolism, which we still know and use today, was introduced by Descartes. It was precisely from then on, thanks to him and Fermat, that algebra merged with geometry. In 1637 Descartes introduced analytical geometry and stated that the 2nd degree equations were precisely the conics studied by Apollonius (262-190 BC). With the introduction of the new algebraic methods linked to the Cartesian plane, Descartes and Fermat managed to solve the problems faced up to them in a simpler way and to verify the properties of conics. In reality the latter ones had previously been discovered by Menaechmus (380-320 BC), while he was trying to solve the problem of duplicating the cube.

Descartes-Shutterstock

Menaechmus was the first to demonstrate that conics could be obtained by intersecting a cone with a plane (hence the name conics). Later Apollonius demonstrated that they could also be obtained from a double-pitched cone, intersecting it with a plane whose inclination varied. We also owe the names of parabola, ellipse and hyperbola to him. After Apollonius the study of conics was abandoned.

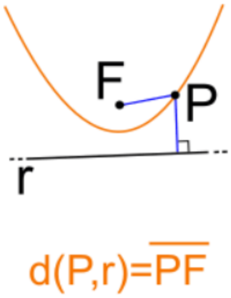


They appeared again during the Renaissance, in art, with perspective, and then in the Baroque, where curved lines (ellipses) took on a privileged place.

Links between these elements and math topics

Conics are of four types: parabola, circumference, ellipse and hyperbola. Two important applications to real situations, for finding maximum and minimum or for acoustic problems, specifically concern parabolas and ellipses on which we will focus.

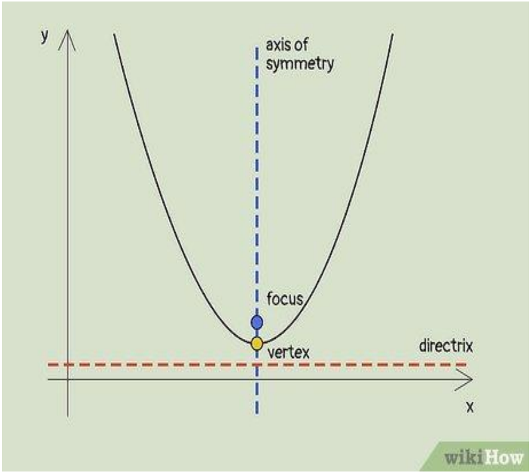
The parabola is the geometric locus of the points on the plane P equidistant from a given point, called focus, F, and from a given straight line, r, called directrix.



From the definition, we can obtain, with some algebraic steps, the equation of the parabola:

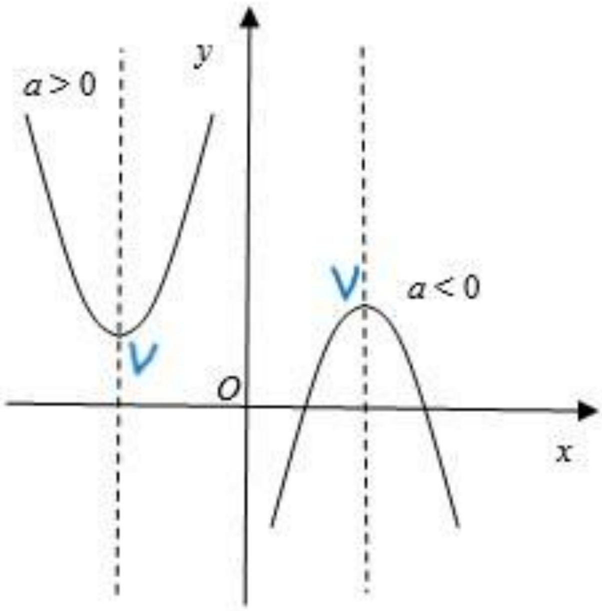
$$y = ax^2 + bx + c$$

Parabolas with this equation have the axis of symmetry parallel to the y axis:



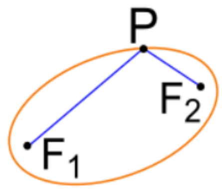
If a>0 the concavity faces upwards and the vertex V is the lowest point.

If a<0 the concavity faces downwards and the vertex V is the highest point.



$$V = \left(-\frac{b}{2a}; -\frac{b^2 - 4ac}{4a}\right)$$

The ellipse is the geometric locus of points P on the plane such that the sum of the distances of these points from two fixed points, called foci, F_1 and F_2, is constant.



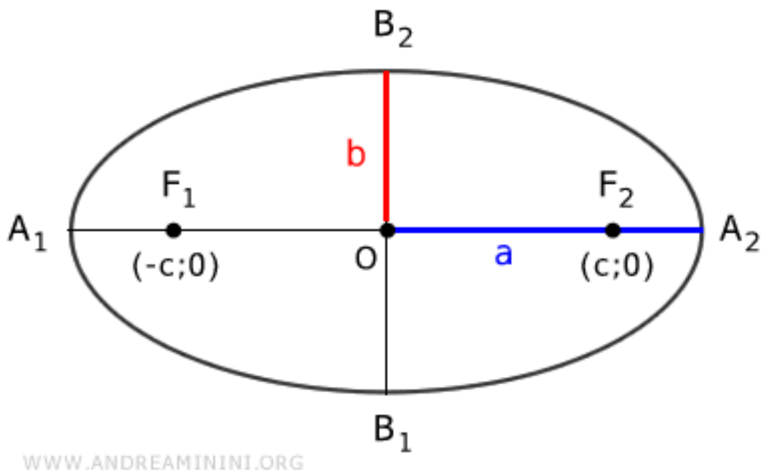
$\overline{PF_1}+\overline{PF_2}=$ constant

From the definition, with some algebraic steps, we arrive at the equation of the ellipse in normal form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

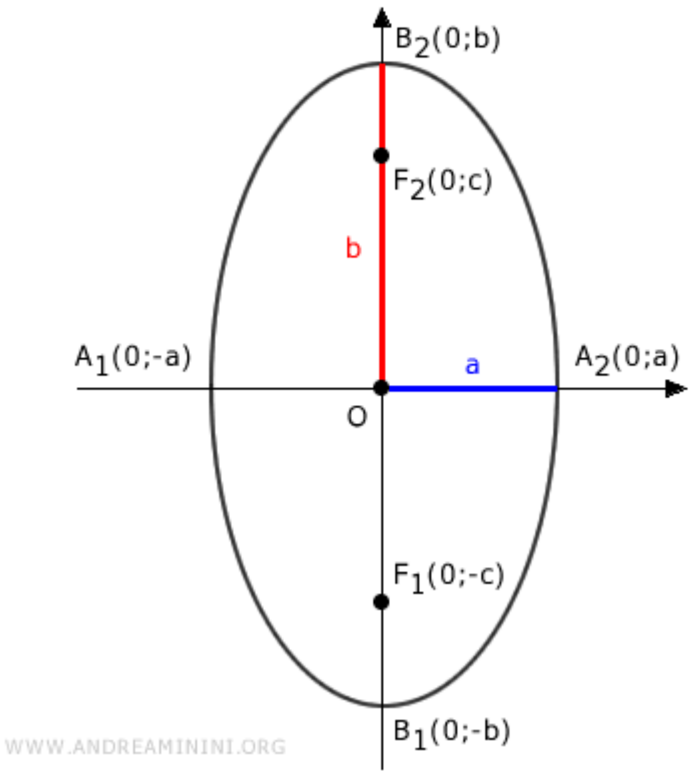
a and b are the lengths of the semi-axes.

If a>b, foci are on x-axis:



$$c=\sqrt{a^2-b^2}$$

If a<b, foci are on y-axis:



$$c = \sqrt{b^2 - a^2}$$

Step 2: Class activities

Exercise 1

Challenge: You only have 320m of fence and want to fence off as much land as you can. The enclosure must be rectangular. How long should the sides of this rectangle be?

If you take L_1=10m and L_2=150m, the rectangle will have area…

If you take L_1=20m and L_2=140m, the rectangle will have area…

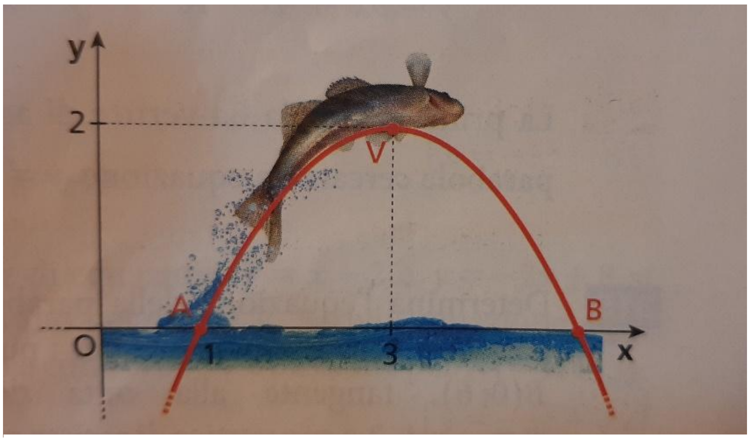
Attention, remember that the perimeter must be 320m: $2L_1+2L_2$=320m, therefore L_1+L_2=160m.

After trial and error, try to solve the same problem using x as an unknown for one side (and consequently the other will be 160-x) and obtaining the parabola associated with the area formula: y=x(160-x). What special point in this parabola corresponds to the solution to the problem?

Exercise 2

Observe the trajectory of the fish. Do you recognize any particular curve? What does the highest point of this curve represent? Determine the coordinates of this point. Then find the abscissa of point B, where the fish re-enters the water.

*Challenge: try to determine the equation of the curve that describes the trajectory of the fish.

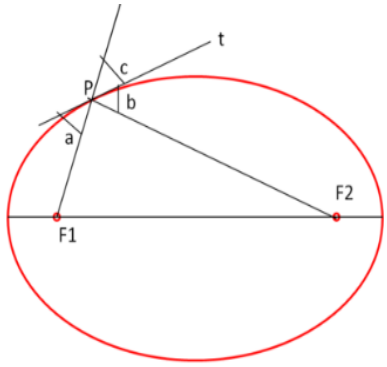


Exercise 3

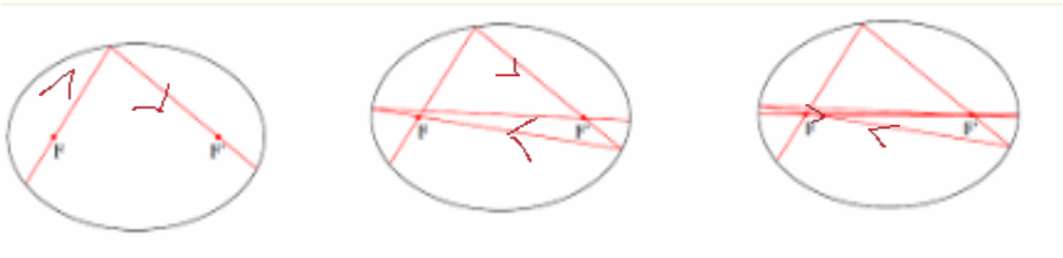
Try to construct an ellipse based on the characteristics of the geometric place. Take a sheet of paper, pierce it in two places - which will correspond to the two foci - and cut a piece of string of the length you want. Attach the thread to the two points, securing it on the back with the adhesive tape, and then with a pencil, pulling the string until it is taut, draw the ellipse on the sheet. At the following link you can watch a video that shows it:

https://www.youtube.com/shorts/nKqfHrYFne8

Trace a trajectory of a billiard ball on the drawn ellipse and calculate the trajectory of the bounce on the ellipse, knowing that the bounce has the same angle with respect to the perpendicular of the arrival trajectory.



Where do the balls that start from a focus, after the first bounce, definitely arrive? Use the protractor to accurately calculate the angle.



Exercise 4

In this photo of the Colosseum you can see the elliptical-shaped structure. The maximum length (major axis) of the amphitheatre is approximately 188m, while its maximum width (minor axis) is approximately 156m. Obtain the equation of the ellipse that represents the external contour of the building, choosing the centre of the ellipse as the origin of the reference system and the straight line containing the major axis as the x-axis. Where are the foci in this reference system?

Bergamini-Barozzi-Trifone-Lineamenti di matematica.azzurro-vol.3

Step 3: Homework and development ideas

Exercise 1

A 20cm long thread is cut into two parts. With the two pieces obtained, two squares are formed. At what point should we cut the thread so that the sum of the squares is minimal?

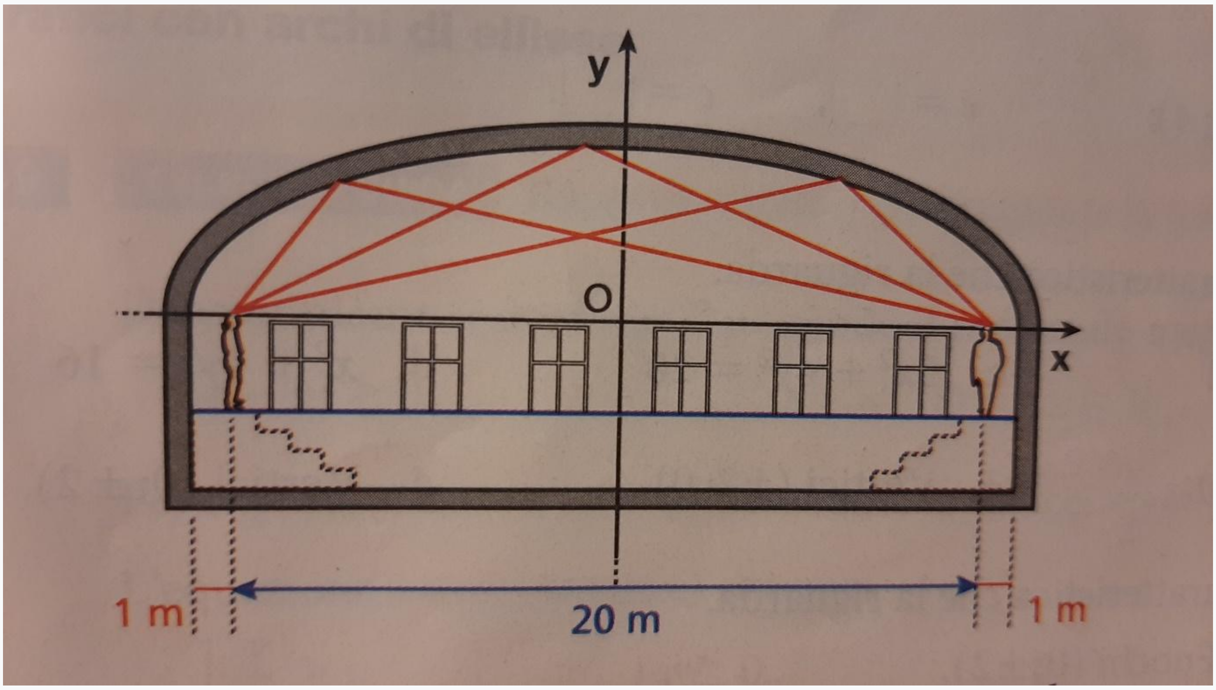
Exercise 2

A window is formed by a rectangle surmounted by a semicircle, with the diameter coinciding with one side of the rectangle. If the perimeter of the window is 24m, determine the radius of the semicircle so that the light entering through the window is maximum.

Exercise 3

An ellipsoidal vaulted chamber has the acoustic property that, if two people position themselves in the foci of the ellipse and speak in a low voice, with their backs turned, they can hear each other very well, as if they were close. This is due to the reflection of the acoustic waves on the walls of the vault. In the following figure we see two people with their heads in the foci of the ellipse. Find the equation.

(From the drawing you can obtain the length of the major axis, 2a, and the focal distance, 2c. How you can get b?)



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Material needed for the tour

Twine, sheets, scissors, tape, protractor, pencil.

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