Arithmetic Sequence and Geometric Sequence

Topic	Number Sequences
Learning objectives	Application of number sequences in practical calculations
Age group	14-18 years (to be adapted in each country)
Estimated duration	2 hours
Activities	Using number sequences for financial calculations
Related visits	Warsaw, Lille, Amiens, Lucca, Agen

Previous knowledge required

The student should be able to:

- calculate the next term of an arithmetic sequence given its first term and the common difference;
- calculate the next term of a geometric sequence given its first term and the common ratio;
- apply formulas for the sum of the first n terms for both sequences.

Step by step: the sequence in the classroom

Step 1: Introducing the topic

Short presentation of the heritage elements in this sequence

An interesting fact related to arithmetic sequences comes from the life of Carl Friedrich Gauss (1777 - 1855), a German scientist and inventor. Gauss showed exceptional mathematical abilities from a young age.

Supposedly, when Gauss was 7 years old, his teacher asked the students to calculate the sum of natural numbers from 1 to 50 (which form an arithmetic sequence with the first term and common difference both equal to 1). The teacher likely hoped this task would take the students a long time, providing him peace for the rest of the class. Imagine his surprise when young Gauss provided the correct answer in no time. Perhaps Gauss calculated it as follows: he wrote down numbers from 1 to 50, and beneath them, the same numbers in reverse order, then added them up:

$$\begin{array}{r} 1 + 2 + 3 + 4 + \ldots + 49 + 50 \\ + \quad 50 + 49 + 48 + 47 + \ldots + 2 + 1 \\ \hline 51 + 51 + 51 + 51 + \ldots + 51 + 51 = 51 * 50 \end{array}$$

This way, he obtained a number twice as large as the sought sum. Therefore:

$$1 + 2 + 3 + \ldots + 49 + 50 = \frac{51 * 50}{2} = 1275$$

Links between these elements and math topics

The method Gauss used to calculate the sum of the first 50 natural numbers can be applied to find the sum of the first n terms of any arithmetic sequence:

$$S_n = \frac{a_1 + a_n}{2} \cdot n$$

S_n- sum of the first n terms of the sequence

a_1- the first term of the sequence

a_n- the last term of the sequence

In the case of what Gauss calculated, we can say that the numerator in this formula:

$a_1+a_n=51$, while $n=50$.

By the way, let's recall the formula for the next term of an arithmetic sequence:

$$a_n = a_1 + (n-1) \cdot r$$

Where ***r*** is the common difference.

A geometric sequence, on the other hand, is a numerical sequence with at least three terms, where each term starting from the second is obtained by multiplying the previous term by a number called the common ratio.

An example of such a sequence is: -1, 2, -4, 8, ... where the first term $a_1 = -1$ and the common ratio (q) is (-2).

To obtain the next terms of a geometric sequence, we need to use the formula:

$$a_n = a_1 \cdot q^{n-1}$$

The formula for the sum of the first n terms of a geometric sequence is:

$$S_n = a_1 \cdot \frac{1 - q^n}{1 - q}$$

S_n- sum of the first n terms of the sequence

a_1- the first term of the sequence

q - the common ratio

Step 2: Class activities

Warm-up

1. In an arithmetic sequence given $a_1 = 16$ and r=-2. Calculate a_{50} and the sum of the first 50 terms of this sequence.
2. Check if the numbers: $14,\ 2,\ \frac{2}{7},\ \frac{1}{2}$ form a geometric sequence.

Pocket Money

Imagine an incredibly wealthy relative visits you and promises to give you pocket money every month for the next two years. They present you with two options:

Option 1:	Option 2:
In the first month, you receive **€100**, and in each subsequent month, you will receive more than in the previous month by an amount equal to 50% of the pocket money in the first month.	In the first month, you will receive **€2**, and in each subsequent month, you will receive 50% more than in the previous month.

Which option would you choose?

Let's calculate how much pocket money you would receive in each month. K_n denotes pocket money (in €) received in the n-th month. So, the situation looks like this:

Option 1:

$$K_1 = 100$$
$$K_2 = K_1 + 50 = 100 + 50$$
$$K_3 = K_2 + 50 = 100 + 2 * 50$$
$$K_4 = K_3 + 50 = 100 + 3 * 50$$
$$\dots$$
$$K_n = K_{n-1} + 50 = 100 + (n-1) * 50$$

Option 2:

$$K_1 = 2$$
$$K_2 = K_1 * 1{,}5 = 2 * 1{,}5$$
$$K_3 = K_2 * 1{,}5 = 2 * 1{,}5^2$$
$$K_4 = K_3 * 1{,}5 = 2 * 1{,}5^3$$
$$\dots$$
$$K_n = K_{n-1} * 1{,}5 = 2 * 1{,}5^{n-1}$$

The successive payments in the 1st option form an arithmetic sequence (with the first term K_1=100 and the common difference r=50), and in the 2nd option - a geometric sequence (with the first term K_1=2 and the common ratio q=1.5).

Exercise 1

Complete the table, and you will find out how much pocket money you would receive in each month of the first year:

Month	1	2	3	4	5	6	7	8	9	10	11	12
Option 1	100	150	200	?	?	?	?	?	?	?	?	650
Option 2	3	4.5	6.75	?	?	?	?	?	?	?	?	173

This table shows how much pocket money you would receive in the last month of the first and second years:

	Option 1:	Option 2:
K_{12}	100+11*50=650	$2*1,5^{11}\approx 173$
K_{24}	100+23*50=1 250	$2*1,5^{23}\approx 22\,445$

If we choose option 1, we would receive more in the last month of the first year than option 2. However, in the last month of the second year, the payout according to option 2 is significantly higher.

Now let's calculate the sum of all payments from both options. Here, we will use formulas for the sum of the first n terms of an arithmetic and geometric sequence.

Option 1:

$$S_{24}=\frac{K_1+K_{24}}{2}*24=$$
$$=\frac{100+1250}{2}*24=$$
$$=22\,800$$

Option 2 :

$$S_{24}=K_1*\frac{1-q^{24}}{1-q}=$$
$$=2*\frac{1-(1,5)^{24}}{1-1,5}\approx$$
$$\approx 67\,332$$

Option 2, although initially seemed less attractive, turned out to be much more beneficial than option 1.

Both these options remind us of the two most common ways of calculating interest in various financial operations. Let's take a closer look at option 2.

In this method, interest is calculated on the capital increased by interest from the previous period. If the initial capital was K_0 and the interest rate is p%, then after each interest calculation period, the capital along with interest is:

After the first period: $K_1=K_0+\frac{p}{100}K_0=K_0\left(1+\frac{p}{100}\right)$

After two periods: $$K_2 = K_1 + \frac{p}{100}K_1 = K_1\left(1 + \frac{p}{100}\right) = K_1\left(1 + \frac{p}{100}\right)^2$$

After n periods: $$K_n = K_0\left(1 + \frac{p}{100}\right)^n$$

This method of calculating interest (profit) is usually applied when calculating interest on bank deposits. When the bank adds the interest amount to the capital (account balance), we say that the interest is compounded.

Note: In the following tasks, when we talk about interest on bank deposits, we do not consider interest (profit) tax, i.e., we assume that the interest rate provided is net.

Task 1

On an account with an interest rate of 4% per annum, €10,000 was deposited. Calculate the account balance after 5 years if the account owner makes no deposits or withdrawals.

K_0 = € 10 000	**K_0 - amount deposited**
p% = 4%	p - annual interest rate
n = 5	n - number of years (interest accrual periods)
	K_5 - account balance after 5 years

$$K_5 = K_0\left(1 + \frac{p}{100}\right)^5 = 10\,000\left(1 + \frac{4}{100}\right)^5 = ?$$

Complete the calculations. Is the profit from this deposit higher than €2 000?

For deposits shorter than 1 year, the bank compounds interest after each period of the deposit's duration. For example, if we leave money in a 3-month deposit for 2 years, interest will be calculated every 3 months, in this case, 8 times. We should also consider this in determining the interest rate (p). We will discuss this in the next task.

Task 2

We deposit €30,000 in a 2-month deposit. The interest rate for this deposit is 5% per annum. What will be the account balance after one and a half years?

K_0 = € 30 000

n = 9 (1.5 years = 18 months, and the deposit is 2-month)

$$p\% = \frac{2}{12} * 5\% = \frac{5}{6}\%$$

Calculate the interest rate for the deposit period.

Calculate according to the formula: $K_n = K_0\left(1 + \frac{p}{100}\right)^n$ account balance after 18 months:

$$K_9 = 30\,000\left(1 + \frac{5}{6 * 100}\right)^9 = ?$$

Step 3: Homework and development ideas

Karol, who is 25 years old, intends to deposit €100 into a retirement fund at the beginning of each month. The fund's interest rate is 6% per annum, and interest is compounded monthly. What capital will Karol accumulate when he reaches the age of 65?

Materials needed for the tour

Calculator

VISIT MATH

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